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Gauge/Gravity Duality for Interactions of Spherical Membranes in 11-dimensional pp-wave

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Abstract

We investigate the gauge/gravity duality in the interaction between two spherical membranes in the 11-dimensional pp-wave background. On the supergravity side, we find the solution to the field equations at locations close to a spherical source membrane, and use it to obtain the light cone Lagrangian of a spherical probe membrane very close to the source, i.e., with their separation much smaller than their radii. On the gauge theory side, using the BMN matrix model, we compute the one-loop effective potential between two membrane fuzzy spheres. Perfect agreement is found between the two sides. Moreover, the one-loop effective potential we obtain on the gauge theory side is valid beyond the small-separation approximation, giving the full interpolation between interactions of membrane-like objects and that of graviton-like objects.

1 Introduction

In this paper we will investigate the gauge/gravity duality in the maximally supersymmetric eleven-dimensional pp-wave background. The gauge theory side is represented by a 1d matrix theory first proposed in [1] while the other side is represented by 11d supergravity. Similar investigations have been performed in flat space for various M-theory objects [3, 4, 5, 6, 7, 8, 9]. In particular, we will be interested in the two-body interactions of point like gravitons and spherical M2 branes.

In pp-wave there is an interesting new connection we could make between a graviton and an M2 brane. Under the influence of the 3-form background whose strength is proportional to a parameter μ, each stable M2 brane curls up into a sphere, with its radius r_0 proportional to μ and its total momentum in the x^- direction, i.e. we have $r_0 \sim \mu P_-$. If one takes the limit of $\mu \to 0$ while keeping the total momentum P_- fixed, then the radius of the sphere goes to zero and we get a point like graviton. If on the other hand, we increase P_- simultaneously such that the radius goes to infinity, then the end product will be a flat membrane instead. Thus the gravitons and the flat membranes of flat space could both be regarded as different limit of spherical membranes in pp-wave. One could make another observation by considering two spherical membranes separated by a distance z. If $z \gg r_0$, then one expects the membranes to interact like two point-like gravitons. If $z \ll r_0$ then the interaction should be akin to that between flat membranes. Therefore by computing the interactions between membranes of arbitrary radii and separation, we could then take different limit to understand the interactions of both gravitons and membranes.

In this paper we will compare the light-cone Lagrangian of supergravity with the effective potential of matrix theory. With a slight abuse in terminology, we will sometimes refer to both as the effective potential. On the supergravity side, we will use linear approximation when solving the field equations. This means all the metric components computed this way will be proportional to no higher than the first power of κ_{11}^2, higher order effects such as recoiling and other back reactions could then be neglected. On the matrix theory side, the interaction between two spherical membranes begins at one loop, and for the purpose of comparing to linearized supergravity, only the one-loop effective potential is needed. A more detailed discussion can be found in [2].

Due to the complexity of the field equations of 11d supergravity, we will only compute the effective potential of the supergravity side in the near membrane limit ($z \ll r_0$). The graviton limit ($z \gg r_0$) was already computed in our previous paper [2]. On the matrix theory side, however, it is possible to compute the expression for general z and r_0, and by taking the appropriate limits, we are able to find perfect agreement with supergravity. In

other words, the matrix theory result provides a smooth interpolations between the near membrane limit and the graviton limit. We also compare our results with the those of Shin and Yoshida [10, 11] and where there is overlap we again have perfect agreement.

This paper is organized as follows. We begin in section 2 with a brief discussion of the theories on both sides of the duality. The relations between the parameters on the two sides as well as a discussion on the special limits of interests can be found here. In section 3 we first define the probe membrane's light cone Lagrangian on the supergravity side for a general background, and then consider it in the pp-wave background perturbed by a source. Next in section 4 the linearized field equations are first diagonalized for an arbitrary static source, and then solved in the special case of a spherical membrane source in the near membrane limit ($z \ll r_0$). The metric and the three-form potential are then used to compute the supergravity light cone Lagrangian. Section 5 is devoted to computation on the matrix theory side. The one-loop effective potential is found by integrating over all the fluctuating fields. The near membrane limit of the resulting potential is compared with the supergravity light cone Lagrangian and agreement is found. In section 6 we will compute the one-loop interpolating effective potential on the matrix theory side for arbitrary separation that takes us between the membrane limit and the graviton limit. The result is compared with our earlier work [2] as well as with that of Shin and Yoshida [10, 11]. This is followed by a discussion in section 7.

2 The Two Sides of the Duality

2.1 The Spherical Membranes

The nonzero components of the maximally supersymmetric eleven-dimensional pp-wave metric and the four-form field strength are given by:

$$g_{+-} = 1, \; g_{++} = -\mu^2 \left[\frac{1}{9} \sum_{i=1}^{3} (x^i)^2 + \frac{1}{36} \sum_{a=4}^{9} (x^a)^2 \right], \; g_{AB} = \delta_{AB} \tag{1}$$

$$F_{123+} = \mu \tag{2}$$

The index convention throughout this paper is: $\mu, \nu, \rho, \ldots$ take the values $+, -, 1, \ldots, 9$; $A, B, C, \ldots$ take the values $1, \ldots, 9$; $i, j, k, \ldots$ take the values $1, \ldots, 3$; and $a, b, c, \ldots$ take the values $4, \ldots, 9$.

The matrix theory action in such a background is known [1]:

$$
\begin{aligned}
\mathcal{S} = \int dt Tr \Bigg\{ & \sum_{A=1}^{9} \frac{1}{2R}(D_t X^A)^2 + i\psi^T D_t \psi + \frac{(M^3R)^2}{4R} \sum_{A,B=1}^{9} [X^A, X^B]^2 \\
& -(M^3R) \sum_{A=1}^{9} \psi^T \gamma^A [\psi, X^A] + \frac{1}{2R} \left[-(\frac{\mu}{3})^2 \sum_{i=1}^{3} (X^i)^2 - (\frac{\mu}{6})^2 \sum_{a=4}^{9} (X^a)^2 \right] \\
& -i\frac{\mu}{4}\psi^T \gamma_{123}\psi - \frac{(M^3R)\mu}{3R} i \sum_{i,j,k=1}^{3} \epsilon_{ijk} X^i X^j X^k \Bigg\}
\end{aligned} \tag{3}
$$

where $D_t X = \partial_t X^I - i[X_0, X^I]$. The constant M in the action above is the eleven-dimensional Planck mass, and R is the compactification radius in the X^- light-like direction in the DLCQ formalism.

The supersymmetric configurations of this action have been well studied, see for example [12]. Although in the end we will add a perturbation in the X^4 to X^9 directions to make it non-supersymmetric, we begin with the following configuration:

$$
\begin{aligned}
X_i &= \frac{\mu}{3MR^3} J_i \\
X_a &= 0
\end{aligned} \tag{4}
$$

where $[J_i, J_j] = i\epsilon_{ijk} J_k$.

If J_i is an $N \times N$ irreducible matrix, then it represents a single sphere with radius r_0 given by:

$$
r_0 = \sqrt{\frac{1}{N} Tr X_i^2} = \frac{\mu}{6M^3R}\sqrt{N^2-1} \approx \frac{\mu N}{6M^3R} \tag{5}
$$

The last approximation is taken with N large, such that the matrix theory correction to the radius (denoted as δr_0) is negligible compared to r_0. In other words, we choose N large enough that $\frac{\delta r_0}{r_0} \sim \frac{1}{N^2} \to 0$. The reader is reminded that the purpose of this paper is to compare matrix theory to the predictions of supergravity, so we are not interested in any finite N effects which are related to matrix theory corrections to supergravity.

If J_i is reducible it represents multiple concentric spherical membranes, and the radius of each irreducible component can be found in the same way as for the case above.

The configuration we are going to use is one where J_i is the direct sum of two irreducible components. This represents two membranes of different radius. In section 5 we will put in non-trivial X_a to break the supersymmetry, which physically corresponds to placing one of the membranes away from the origin.

On the supergravity side, the bosonic part of the Lagrangian of a probe membrane in a general background is

$$
\mathcal{L}(X^\mu, \partial_i X^\mu) = -T \left[\sqrt{-\det(g_{ij})} - \frac{1}{6}\epsilon^{ijk} A_{\mu\nu\rho} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho \right] \tag{6}
$$

with T being the membrane tension, and $g_{ij} \equiv G_{\mu\nu}\partial_i X^\mu \partial_j X^\nu$ being the pullback metric.

One can define the light cone Lagrangian $\mathcal{L}_{lc}$ via a Legendre transformation in the X^- direction (see Section 3). In the unperturbed pp-wave background, we have

$$\begin{aligned}(\mathcal{L}_{lc})_{pp} \quad &= \frac{1}{2}\Pi_-\partial_0 X^A \partial_0 X^A - \frac{\Pi_-}{2}\mu^2\left[\frac{1}{9}(X^i)^2 + \frac{1}{36}(X^a)^2\right] \\ &-\frac{T^2}{4\Pi_-}(\partial_1 X^A \partial_2 X^B - \partial_1 X^B \partial_2 X^A)^2 + T\frac{\mu}{3}\epsilon_{ijk}(\partial_1 X^i)(\partial_2 X^j)X^k \end{aligned} \tag{7}$$

with Π_- being the momentum density in the X^- direction. Looking at $(\mathcal{L}_{lc})_{pp}$, above, wee see that one solution to the equations of motion in the unperturbed pp-wave is a spherical membrane at rest with $X^i X^i = r_0^2$, $X^a = 0$, $X^+ = t$, $X^- = 0$, with its radius being $r_0 = \frac{\mu\Pi_-}{3T\sin\theta}$ (note that we take the worldvolume coordinates $\sigma^{0,1,2}$ to be t, θ, ϕ; also recall that it is $\frac{\Pi_-}{\sin\theta}$ that is a constant on the worldvolume). In terms of the total momentum $P_- = \int d\theta d\phi \Pi_-$, this radius is

$$r_0 = \frac{\mu P_-}{12\pi T} \tag{8}$$

Later in this paper, we will take the background to be the pp-wave perturbed by a source membrane: $G_{\mu\nu} = (G_{\mu\nu})_{pp} + h_{\mu\nu}$, and $A_{\mu\nu\rho} = (A_{\mu\nu\rho})_{pp} + a_{\mu\nu\rho}$. Then $\mathcal{L}_{lc} = (\mathcal{L}_{lc})_{pp} + \delta\mathcal{L}_{lc}$, and we shall compare $\delta\mathcal{L}_{lc}$ with the one-loop effective potential on the matrix theory side (while $(\mathcal{L}_{lc})_{pp}$ agrees with the tree level Lagrangian on the matrix theory side, of course).

Identifying the radius calculated on the two sides and P_- with N/R, we get the following relation between the tension of the membrane and the Planck mass:

$$2\pi T = M^3 \tag{9}$$

Another relation we will use to compare the two side is the identification $\kappa_{11}^2 = \frac{16\pi^5}{M^9}$ [2, 13], and for convenience we will define the parameter $\alpha = \frac{1}{M^3 R}$ that will appear often on the matrix theory side.

2.2 The Effective Potential

As already stated, on the supergravity side the quantity of interest is the probe light cone Lagrangian. On the matrix theory side, the one-loop effective potential is the relevant quantity, and is defined through the Euclideanized effective action with all quadratic fluctuations integrated out. In the following, the subscripts $_M$ and $_E$ will be used to denote Minkowski and Euclidean signature respectively.

Euclideanization is carried out by defining the following Euclideanized quantities:

$$\begin{aligned}\tau_E &= i\tau_M \\ \mathcal{S}_E &= -i\mathcal{S}_M\end{aligned} \tag{10}$$

The Euclideanized action $\mathcal{S}_E$ will then be expanded about a certain background (to be specified in section 5) and the quadratic fluctuations integrated out to produce the one-loop effective action Γ_E. The effective action on the matrix theory side is then defined via the relation:

$$\Gamma_E = -\int d\tau_E V_{eff} \tag{11}$$

The minus sign in front of the integral is slightly unconventional, but it was put there for the convenience of comparison with supergravity. It was chosen such that the tree level part of the effective potential is simply the light cone Lagrangian $(\mathcal{L}_{lc})_{pp}$ rather than $-(\mathcal{L}_{lc})_{pp}$. After V_{eff} is computed, the result could then be analytically continued back into Minkowski signature by replacing $v_E \rightarrow iv_M$.

In order to facilitate the comparison of the two sides, it is useful to first examine the form of the action. The supergravity effective action is given in eqn(95). For the simple case when the two membranes have the same radius and in the limit where their separation in the X^4 to X^9 direction, z, is small, i.e. $z \ll r_0$, the supergravity effective action is given in eqn(95). Putting w, the difference of the two radii to zero, and not keeping track of the exact coefficients, we could rewrite the supergravity result in term of matrix theory parameters:

$$V_{eff} = \alpha\left(\frac{v^4}{\mu^2 z^5} + \frac{v^2}{z^3} + \frac{\mu^2}{z}\right) \tag{12}$$

Hence we see that a comparison with supergravity means looking at order $(\alpha)^1$ on the matrix theory side.

2.3 The Membrane Limit and the Graviton Limit

We will first state the two limits we are interested in:

Membrane limit:

$$\frac{z}{\alpha\mu} \gg 1 \tag{13}$$

$$\frac{z}{\alpha\mu} \ll N \tag{14}$$

Graviton limit:

$$\frac{z}{\alpha\mu} \gg 1 \tag{15}$$

$$\frac{z}{\alpha\mu} \gg N \tag{16}$$

where we used z to denote the separation of the two spherical membranes in the X^4 to X^9 directions, and recall $\alpha = \frac{1}{M^3 R}$.

The membrane limit is derived from the condition:

$$\frac{1}{N} \ll \frac{z}{r_0} \ll 1 \tag{17}$$

The first inequality ensures that the effect of non-zero z is greater than any matrix theory corrections to supergravity, which we are not interested in. The second inequality ensures we are at the near membrane limit. Using $r_0 = \frac{\alpha \mu N}{6}$ (see eqn (5)), we arrive at the limit as stated.

The graviton limit is when z is much greater than r_0, so that approximately the two spheres interact like two point like gravitons. We still enforce the condition $\frac{1}{N} \ll \frac{z}{r_0}$ for comparison with supergravity but reverse the second inequality in the membrane limit to $\frac{z}{r_0} \gg 1$ to arrive at the graviton limit stated above.

In this paper, on the supergravity side we will calculate the light cone Lagrangian only in the membrane limit. The light cone Lagrangian in the graviton limit was already computed in our earlier work [2]. On the matrix theory side, the one-loop effective potential could be computed for general z. The two limits of this potential is then compared with the supergravity side and we will find perfect agreement. Later in the paper we will use matrix theory to find explicitly the potential that interpolates between the two limits.

3 The Supergravity Light Cone Lagrangian

The light cone Lagrangian $\mathcal{L}_{lc}$ of the probe is basically its Lagrangian Legendre transformed in the x^- degree of freedom. Here we briefly derive the $\mathcal{L}_{lc}$ for a probe membrane in a pp-wave background perturbed by some source. (The reader is referred to Section 4.1 of [14] for the detailed discussion of the light-cone Lagrangian of point particle probes and membrane probes in terms of Hamiltonian systems with constraints.)

Denote the background metric and three-form as $G_{\mu\nu}(x)$, $A_{\mu\nu\rho}(x)$, respectively, and the membrane embedding coordinates as $X^\mu(\sigma^i)$, with $\sigma^i, i = 0, 1, 2$ being the world-volume coordinates. The bosonic part of the membrane Lagrangian density is given by

$$\mathcal{L}(X^\mu, \partial_i X^\mu) = -T \left[\sqrt{-\det(g_{ij})} - \frac{1}{6} \epsilon^{ijk} A_{\mu\nu\rho} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho \right] \tag{18}$$

with T being the membrane tension, and $g_{ij} \equiv G_{\mu\nu} \partial_i X^\mu \partial_j X^\nu$ being the pullback metric. The momentum density is

$$\Pi_\lambda \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 X^\lambda)} = -T \left[\sqrt{-\det(g_{ij})}\; g^{0k} (\partial_k X^\mu) G_{\lambda\mu} - A_{\lambda\nu\rho} \partial_1 X^\nu \partial_2 X^\rho \right] \tag{19}$$

Define

$$\tilde{\Pi}_\lambda \equiv \Pi_\lambda - T A_{\lambda\nu\rho}\partial_1 X^\nu \partial_2 X^\rho = -T\sqrt{-\det(g_{ij})}\; g^{0k}(\partial_k X^\mu) G_{\lambda\mu} \tag{20}$$

There are the following three primary constraints

$$\begin{aligned} \phi_0 &\equiv G^{\lambda\xi}\tilde{\Pi}_\lambda \tilde{\Pi}_\xi + T^2 \det(g_{rs}) = 0 \\ \phi_r &\equiv \Pi_\lambda \partial_r X^\lambda \end{aligned} \tag{21}$$

where $r, s = 1, 2$ label the spatial world-volume coordinates.

Now use the light-cone coordinates $\{x^+, x^-, x^A\}$, and assume static, i.e., x^+-independent, background $G_{\mu\nu}(x^-, x^A)$ and $A_{\mu\nu\rho}(x^-, x^A)$. Use the light cone gauge $X^+ = \sigma^0$. Then the equation of motion for X^+ fixes the Lagrange multiplier c^0 for the constraint ϕ_0

$$c^0 = \frac{1}{2G^{+\xi}\tilde{\Pi}_\xi} \tag{22}$$

The constraint $\phi_0 = 0$ (in which X^+ is set to σ^0) can be used to solve for Π_+ as a function of $(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \Pi_A)$.

We shall also express Π_A as a function of $(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \partial_0 X^A)$ by inverting the equations of motion for X^A:

$$\frac{\partial X^A}{\partial \sigma^0} = \frac{G^{A\xi}\tilde{\Pi}_\xi}{G^{+\nu}\tilde{\Pi}_\nu} + c^r \frac{\partial X^A}{\partial \sigma^r} \tag{23}$$

where we have used the expression for c^0 given above and the c^r's are the Lagrange multipliers for the ϕ^r's.

Then we can first define the light-cone Hamiltonian $\mathcal{H}_{lc}(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \Pi_A) \equiv -\Pi_+$, and then define the light-cone Lagrangian

$$\mathcal{L}_{lc}(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \partial_0 X^A) \equiv \Pi_A \partial_0 X^A - \mathcal{H}_{lc} = \Pi_A \partial_0 X^A + \Pi_+ \tag{24}$$

Now the background in which the probe membrane moves is the pp-wave perturbed by some source: $G_{\mu\nu} = (G_{\mu\nu})_{pp} + h_{\mu\nu}$, and $A_{\mu\nu\rho} = (A_{\mu\nu\rho})_{pp} + a_{\mu\nu\rho}$, where the quantities with subscript pp are those of the unperturbed pp-wave background, and $h_{\mu\nu}, a_{\mu\nu\rho}$ are the perturbations caused by the source. We only need the light-cone Lagrangian to linear order in the perturbation. Solving $\phi_0 = 0$ gives

$$\Pi_+(X^-, X^A; \partial_r X^-, \partial_r X^A; \Pi_-, \Pi_A) = (\Pi_+)_{pp} + \delta\Pi_+ \tag{25}$$

where

$$\begin{aligned} (\Pi_+)_{pp} = \quad & \frac{-1}{2\Pi_-}\left\{ -g_{++}\Pi_-^2 + \Pi_A \Pi_A + \frac{T^2}{2}(\partial_1 X^A \partial_2 X^B - \partial_1 X^B \partial_2 X^A)^2 \right\} \\ & + T\frac{\mu}{3}\epsilon_{ijk}(\partial_1 X^i)(\partial_2 X^j) X^k \end{aligned} \tag{26}$$

and

$$
\begin{aligned}
\delta\Pi_+ = \frac{-1}{2\Pi_-}\Bigg\{ \quad & -h_{--}(\tilde{\Pi}_+)^2_{pp} - 2\Pi_-(h_{+-} - g_{++}h_{--})(\tilde{\Pi}_+)_{pp} - 2T(\tilde{\Pi}_+)_{pp}a_{-\nu\rho}\partial_1 X^\nu \partial_2 X^\rho \\
& -2T\Pi_- a_{+\nu\rho}\partial_1 X^\nu \partial_2 X^\rho - 2h_{-A}\Pi_A(\tilde{\Pi}_+)_{pp} - \Pi_-^2[h_{++} + (g_{++})^2 h_{--} - 2g_{++}h_{+-}] \\
& +2T\Pi_- g_{++}a_{-\nu\rho}\partial_1 X^\nu \partial_2 X^\rho - h_{AB}\Pi_A\Pi_B - 2\Pi_-(h_{+A} - g_{++}h_{-A})\Pi_A \\
& -2T\Pi_A a_{A\nu\rho}\partial_1 X^\nu \partial_2 X^\rho + T^2\delta\det(g_{rs}) \Bigg\}
\end{aligned}
\tag{27}
$$

where

$$
(\tilde{\Pi}_+)_{pp} = \frac{-1}{2\Pi_-}\left\{ -g_{++}\Pi_-^2 + \Pi_A\Pi_A + \frac{T^2}{2}(\partial_1 X^A \partial_2 X^B - \partial_1 X^B \partial_2 X^A)^2 \right\} \tag{28}
$$

The Lagrange multipliers c^r's are

$$
c^r = (c^r)_{pp} + \delta c^r = \delta c^r \tag{29}
$$

where we have taken $(c^r)_{pp}$ to be zero. Inverting eqn (23) gives

$$
\Pi_A(X^-, X^A; \partial_r X^-, \partial_r X^A; \Pi_-, \partial_0 X^A) = (\Pi_A)_{pp} + \delta\Pi_A \tag{30}
$$

where $(\Pi_A)_{pp} = \Pi_-\partial_0 X^A$, and $\delta\Pi_A$ depends on δc^r.

Now

$$
\mathcal{L}_{lc} = (\Pi_A)_{pp}\partial_0 X^A + \delta\Pi_A\partial_0 X^A + (\Pi_+)_{pp} + \delta\Pi_+ \tag{31}
$$

However, as can be seen easily, the second term in the above expression is cancelled by the part containing $\delta\Pi_A$ in the third term. So $\mathcal{L}_{lc}$ does not contain $\delta\Pi_A$, thus being independent of the δc^r's.

Hence one finds

$$
\mathcal{L}_{lc} = (\mathcal{L}_{lc})_{pp} + \delta\mathcal{L}_{lc} \tag{32}
$$

with

$$
\begin{aligned}
(\mathcal{L}_{lc})_{pp} \quad &= \frac{1}{2}\Pi_-\partial_0 X^A \partial_0 X^A - \frac{\Pi_-}{2}\mu^2\left[\frac{1}{9}(X^i)^2 + \frac{1}{36}(X^a)^2\right] \\
& -\frac{T^2}{4\Pi_-}(\partial_1 X^A \partial_2 X^B - \partial_1 X^B \partial_2 X^A)^2 + T\frac{\mu}{3}\epsilon_{ijk}(\partial_1 X^i)(\partial_2 X^j)X^k
\end{aligned}
\tag{33}
$$

and $\delta\mathcal{L}_{lc} = \delta\Pi_+$, where $\delta\Pi_+$ is given by eqn (27) with all the Π_A's in it replaced by $(\Pi_A)_{pp} = \Pi_-\partial_0 X^A$ (the reason being that $\delta\Pi_+$ contains $h_{\mu\nu}$ and $a_{\mu\nu\rho}$ and is thus already first-order). We'd like to emphasize again that $\mathcal{L}_{lc}$ is independent of the choice of the δc^r's.

To use the general expression for $\mathcal{L}_{lc}$ given in the previous paragraph in a specific problem, one just have to substitute in the $h_{\mu\nu}$'s and $a_{\mu\nu\rho}$'s found by solving the supergravity field equations for the specific source, as well as the specific probe membrane embedding $X^A(\sigma^i)$.

4 Supergravity Computation

4.1 Diagonalizing the Field Equations for Arbitrary Static Source

In this subsection we present the diagonalization of the linearized supergravity equations of motions for arbitrary static sources (this subsection is basically Section 4.2 of [14]). There is, of course, no highbrow knowledge involved here: we are just solving the linearized Einstein equations and Maxwell equations, which are coupled; and by "diagonalization" we basically just mean the prescription by which we get a decoupled Laplace equation for each component of the metric and three-form perturbations. The unperturbed background is the 11-D pp-wave, and we only consider static, i.e., x^+-independent, field configurations, thanks to the fact that the sources considered are taken to be static, i.e., with x^+-independent stress tensor and three-form current.

Since we leave the source arbitrary, what we'll present here are the left-hand side of the linearized equations. These are tensors whose computation is straightforward though a bit tedious: the reason we present them here is because they are necessary when solving the field equations, and to the best of our knowledge have not been explicitly given elsewhere.

A somewhat related problem is the diagonalization of the equations of motion when the source is absent. This requires field configurations with x^+-dependence. One work along this line is [15]. Roughly speaking, borrowing the language of electromagnetism, what's considered in [15] are electromagnetic waves in vacuum, while what we are considering here are electrostatics and magnetostatics for arbitrary static sources.

The nonzero components up to (anti)symmetry of the Christoffel symbol, Riemann tensor, and Ricci tensor of the 11-D pp-wave are

$$\begin{aligned} \Gamma^A_{++} &= -\frac{1}{2}\partial_A g_{++}, \quad \Gamma^-_{+A} = \frac{1}{2}\partial_A g_{++} \\ R_{+A+B} &= -\frac{1}{2}\partial_A\partial_B g_{++}, \quad R_{++} = -\frac{1}{2}\partial_C\partial_C g_{++} \end{aligned} \tag{34}$$

(We usually do not substitute the explicit expression of g_{++}, unless that brings significant simplification to the resulting formula)

Now let's perturb the pp-wave background by adding a source. Denote the metric perturbation $\delta g_{\mu\nu}$ by $h_{\mu\nu}$, and the gauge potential perturbation by $\delta A_{\mu\nu\rho} = a_{\mu\nu\rho}$. $h_{\mu\nu}, a_{\mu\nu\rho}$ are treated as rank-two and rank-three tensors, respectively, the covariant derivative ∇ acting on them is defined using the connection coefficient of the unperturbed pp-wave background, and indices are raised/lowered, traces taken using the background metric $g_{\mu\nu}$.

Let's deal with the Einstein equations first. Define $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h$, where $h \equiv g^{\mu\nu}h_{\mu\nu}$.

Without the source, the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \kappa_{11}^2[T_{\mu\nu}]_A = 0 \tag{35}$$

Recall that the stress tensor of the gauge field is

$$[T_{\mu\nu}]_A = \frac{1}{12\kappa_{11}^2}\left(F_{\mu\lambda\xi\rho}F_\nu{}^{\lambda\xi\rho} - \frac{1}{8}g_{\mu\nu}F^{\rho\sigma\lambda\xi}F_{\rho\sigma\lambda\xi}\right) \tag{36}$$

The source perturbs the Einstein equation to

$$\delta\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - \kappa_{11}^2\delta[T_{\mu\nu}]_A = \kappa_{11}^2[T_{\mu\nu}]_S \tag{37}$$

with $[T_{\mu\nu}]_S$ standing for the stress tensor of the source.

As usual, it helps to proceed in an organized manner, grouping different terms in the above perturbed Einstein equations. One finds, $\delta\left(R_{\mu\rho} - \frac{1}{2}Rg_{\mu\rho}\right) = -\frac{1}{2}\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\rho} + K_{\mu\rho} + Q_{\mu\rho}$, and $\kappa_{11}^2\delta[T_{\mu\nu}]_A = N_{\mu\nu} + L_{\mu\nu}$, where the explicit expressions of the symmetric tensors $\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu}, K_{\mu\nu}, Q_{\mu\nu}$,$N_{\mu\nu}$, and $L_{\mu\nu}$ can be obtained after some work. Their definitions and components are given below [1]

- $\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu}$

$$\begin{aligned}\nabla^\sigma\nabla_\sigma\bar{h}_{++} = \quad & g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{++} + \left[-(\partial_A\partial_A g_{++})\bar{h}_{+-} + \frac{1}{2}(\partial_A g_{++}\partial_A g_{++})\bar{h}_{--}\right] \\ & +2\left[\partial_A g_{++}\partial_-\bar{h}_{+A} - \partial_A g_{++}\partial_A\bar{h}_{+-}\right]\end{aligned} \tag{38}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{+-} = g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+-} - \frac{1}{2}(\partial_A\partial_A g_{++})\bar{h}_{--} + \partial_A g_{++}\partial_-\bar{h}_{-A} - \partial_A g_{++}\partial_A\bar{h}_{--} \tag{39}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{+C} = g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+C} - \frac{1}{2}(\partial_A\partial_A g_{++})\bar{h}_{-C} + \partial_A g_{++}\partial_-\bar{h}_{AC} - \partial_C g_{++}\partial_-\bar{h}_{+-} - \partial_A g_{++}\partial_A\bar{h}_{-C} \tag{40}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{--} = g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{--} \tag{41}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{-C} = g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{-C} - \partial_C g_{++}\partial_-\bar{h}_{--} \tag{42}$$

[1] Notice that ∂_+ will never appear because we only consider the static case; also note $g^{\mu\nu}\partial_\mu\partial_\nu = -g_{++}\partial_-^2 + \partial_A\partial_A$ for static configurations.

$$\nabla^\sigma \nabla_\sigma \bar{h}_{CD} = g^{\mu\nu}\partial_\mu\partial_\nu \bar{h}_{CD} - \partial_C g_{++}\partial_- \bar{h}_{-D} - \partial_D g_{++}\partial_- \bar{h}_{-C} \tag{43}$$

- $K_{\mu\nu}$ Its definition is

$$K_{\mu\rho} \equiv \frac{1}{2}\left(R_\mu{}^\xi \bar{h}_{\xi\rho} + R_\rho{}^\xi \bar{h}_{\xi\mu}\right) + R^\sigma{}_{\mu\rho}{}^\xi \bar{h}_{\sigma\xi} + \frac{1}{2} g_{\mu\rho} R^{\xi\sigma}\bar{h}_{\xi\sigma} - \frac{1}{2} R \bar{h}_{\mu\rho} \tag{44}$$

Its components are given by

$$K_{++} = \left(-\frac{1}{2}\partial_A\partial_A g_{++}\right)\left(\bar{h}_{+-} + \frac{1}{2} g_{++}\bar{h}_{--}\right) + \frac{1}{2}\left(\partial_A\partial_B g_{++}\right)\bar{h}_{AB} \tag{45}$$

$$K_{+-} = \left(-\frac{1}{2}\partial_A\partial_A g_{++}\right)\bar{h}_{--} \tag{46}$$

$$K_{+A} = \left(-\frac{1}{4}\partial_C\partial_C g_{++}\right)\bar{h}_{-A} + \left(-\frac{1}{2}\partial_A\partial_B g_{++}\right)\bar{h}_{-B} \tag{47}$$

$$K_{--} = 0 \tag{48}$$

$$K_{-A} = 0 \tag{49}$$

$$K_{AB} = \frac{1}{2}\left[\partial_A\partial_B g_{++} - \frac{1}{2}\delta_{AB}\partial_C\partial_C g_{++}\right]\bar{h}_{--} \tag{50}$$

- $Q_{\mu\nu}$ Its definition is $Q_{\mu\rho} \equiv \frac{1}{2}(\nabla_\mu q_\rho + \nabla_\rho q_\mu) - \frac{1}{2} g_{\mu\rho}\nabla^\alpha q_\alpha$, where $q_\alpha \equiv \nabla^\beta \bar{h}_{\beta\alpha}$. As one can recognize, $Q_{\mu\rho}$ contains the arbitrariness of making different gauge choices when solving the Einstein equation, where one makes a gauge choice by specifying the q_μ's. The components of $Q_{\mu\rho}$ are

$$\begin{aligned}
&Q_{--} = \partial_- q_-, \quad Q_{-A} = \frac{1}{2}(\partial_- q_A + \partial_A q_-), \quad Q_{-+} = \frac{1}{2}(g_{++}\partial_- q_- - \partial_A q_A) \\
&Q_{AB} = \frac{1}{2}(\partial_A q_B + \partial_B q_A) - \frac{1}{2}\delta_{AB}(\partial_- q_+ - g_{++}\partial_- q_- + \partial_A q_A) \\
&Q_{+A} = \frac{1}{2}\left[\partial_A q_+ - (\partial_A g_{++}) q_-\right] \\
&Q_{++} = \frac{1}{2}(\partial_A g_{++}) q_A - \frac{1}{2} g_{++}(\partial_- q_+ - g_{++}\partial_- q_- + \partial_A q_A)
\end{aligned} \tag{51}$$

- $N_{\mu\nu}$ It is defined to be the part of $\kappa_{11}^2 \delta[T_{\mu\nu}]_A$ that contains only the metric perturbation, but not the three-form gauge potential perturbation. Its components are given

by

$$
\begin{aligned}
N_{++} &= \mu^2\left(\frac{1}{3}\bar{h}_{+-} + \frac{1}{12}g_{++}\bar{h}_{--} - \frac{1}{3}\sum_{i=1}^{3}\bar{h}_{ii} + \frac{1}{6}\sum_{a=4}^{9}\bar{h}_{aa}\right) \\
N_{+-} &= \frac{\mu^2}{4}\bar{h}_{--}, \quad N_{+i} = \frac{\mu^2}{2}\bar{h}_{-i}, \quad N_{+b} = 0 \\
N_{--} &= 0, \quad N_{-i} = 0, \quad N_{-b} = 0 \\
N_{ij} &= -\frac{\mu^2}{4}\delta_{ij}\bar{h}_{--}, \quad N_{ib} = 0, \quad N_{ab} = \frac{\mu^2}{4}\delta_{ab}\bar{h}_{--}
\end{aligned} \tag{52}
$$

- $L_{\mu\nu}$ This is defined to be the part of $\kappa_{11}^2\delta[T_{\mu\nu}]_A$ that contains only the three-form perturbation, but not the metric perturbation. Its components are given by

$$
\begin{aligned}
& L_{++} = \mu\left(\delta F_{123+} - \frac{1}{2}g_{++}\delta F_{123-}\right), \ L_{+-} = 0, \ L_{+i} = \frac{\mu}{4}\epsilon_{ijk}\delta F_{+jk-}, \ L_{+b} = \frac{\mu}{2}\delta F_{123b} \\
& L_{--} = 0, \quad L_{-i} = 0, \quad L_{-b} = 0 \\
& L_{ij} = \frac{\mu}{2}\delta_{ij}\delta F_{123-}, \ L_{ib} = \frac{\mu}{4}\epsilon_{ijk}\delta F_{bjk-}, \ L_{bd} = -\frac{\mu}{2}\delta_{bd}\delta F_{123-}
\end{aligned} \tag{53}
$$

Next let us deal with the Maxwell equation. In the absence of the source, it is

$$
\frac{1}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}\ F^{\lambda\mu_1\mu_2\mu_3}\right) - \frac{\tilde{\eta}}{1152}\frac{\epsilon^{\mu_1\cdots\mu_{11}}}{\sqrt{-g}}F_{\mu_4\ldots\mu_7}F_{\mu_8\ldots\mu_{11}} = 0 \tag{54}
$$

where $\tilde{\eta}$ is either $+1$ or -1 depending on the choice of convention, which one can fix later by requiring the consistency of the conventions for the equations and the solutions under consideration. When the source is present, we add its current $J^{\mu_1\mu_2\mu_3}$ to the left-hand side of the above equation, and get

$$
\delta\left[\frac{1}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}\ F^{\lambda\mu_1\mu_2\mu_3}\right) - \frac{\tilde{\eta}}{1152}\frac{\epsilon^{\mu_1\cdots\mu_{11}}}{\sqrt{-g}}F_{\mu_4\ldots\mu_7}F_{\mu_8\ldots\mu_{11}}\right] = J^{\mu_1\mu_2\mu_3} \tag{55}
$$

We can write the left-hand side of the above equation as the sum of two totally antisymmetric tensors $Z^{\mu_1\mu_2\mu_3} + S^{\mu_1\mu_2\mu_3}$, where $Z^{\mu_1\mu_2\mu_3}$ is defined to be the part that contains the metric perturbation only, and $S^{\mu_1\mu_2\mu_3}$ is defined to be the part that contains the three-form perturbation only. One finds

$$
\begin{aligned}
& Z^{+-i} = \mu\epsilon_{ijk}\partial_j\bar{h}_{-k}, \quad Z^{+-b} = 0, \quad Z^{+ij} = \mu\epsilon_{ijk}(\partial_-\bar{h}_{-k} - \partial_k\bar{h}_{--}), \quad Z^{+ib} = 0, \quad Z^{+bc} = 0 \\
& Z^{-ij} = \mu\epsilon_{ijk}\left[\partial_k\left(\frac{1}{3}\bar{h} - \bar{h}_-{}^- - \sum_{i=1}^{3}\bar{h}_{ii}\right) - \partial_b\bar{h}_{kb}\right], \quad Z^{-ib} = \mu\epsilon_{ijk}\partial_j\bar{h}_{kb}, \quad Z^{-bc} = 0 \\
& Z^{ijk} = -\mu\epsilon_{ijk}\left[\partial_-\left(\frac{1}{3}\bar{h} - \bar{h}_-{}^- - \sum_{i=1}^{3}\bar{h}_{ii}\right) - \partial_b\bar{h}_{-b}\right], \quad Z^{ijb} = \mu\epsilon_{ijk}(\partial_-\bar{h}_{kb} - \partial_k\bar{h}_{-b}) \\
& Z^{ibc} = 0, \quad Z^{bce} = 0
\end{aligned} \tag{56}
$$

and

$$S^{+-A} = g^{\mu\nu}\partial_\mu\partial_\nu a_{-+A} + \partial_B g_{++}\partial_- a_{BA-} - \partial_-(\nabla^\mu a_{\mu+A}) + \partial_A(\nabla^\mu a_{\mu+-}) \tag{57}$$

$$S^{+AB} = g^{\mu\nu}\partial_\mu\partial_\nu a_{-AB} - \partial_-(\nabla^\mu a_{\mu AB}) + \partial_A(\nabla^\mu a_{\mu-B}) - \partial_B(\nabla^\mu a_{\mu-A}) \tag{58}$$

$$\begin{aligned} S^{-AB} = \quad & g^{\mu\nu}\partial_\mu\partial_\nu a_{+AB} - g_{++}S^{+AB} \\ & + \{[(\partial_A g_{++})(\partial_- a_{-+B}) + \partial_A(\nabla^\mu a_{\mu+B}) - \partial_A(a_{EB-}\partial_E g_{++})] - [A \leftrightarrow B]\} \\ & -(\partial_D g_{++})\delta F_{D-AB} - \mu\frac{\tilde{\eta}}{24}\epsilon^{-AB\mu_4\ldots\mu_7 123+}\delta F_{\mu_4\ldots\mu_7} \end{aligned} \tag{59}$$

$$\begin{aligned} S^{ABE} = \quad & g^{\mu\nu}\partial_\mu\partial_\nu a_{ABE} - (\partial_A g_{++})(\partial_- a_{-BE}) - (\partial_B g_{++})(\partial_- a_{-EA}) - (\partial_E g_{++})(\partial_- a_{-AB}) \\ & -\partial_A(\nabla^\mu a_{\mu BE}) - \partial_B(\nabla^\mu a_{\mu EA}) - \partial_E(\nabla^\mu a_{\mu AB}) - \mu\frac{\tilde{\eta}}{24}\epsilon^{ABE\mu_4\ldots\mu_7 123+}\delta F_{\mu_4\ldots\mu_7} \end{aligned} \tag{60}$$

Notice that $S^{\mu_1\mu_2\mu_3}$ contains $\nabla^\mu a_{\mu\rho\lambda}$ and its derivatives. Those terms correspond to the gauge freedom for the three-form gauge potential.

Now that we have collected the expressions for the various tensors, we are ready to diagonalize the field equations. Recall that the Einstein equation is

$$-\frac{1}{2}\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu} + K_{\mu\nu} + Q_{\mu\nu} - N_{\mu\nu} - L_{\mu\nu} = \kappa_{11}^2[T_{\mu\nu}]_S \tag{61}$$

and the Maxwell equation is

$$Z^{\mu_1\mu_2\mu_3} + S^{\mu_1\mu_2\mu_3} = J^{\mu_1\mu_2\mu_3} \tag{62}$$

The right-hand side of these equations is given by specifying the source that we consider (recall that the three-form current J is of order κ_{11}^2), hence we only need to concentrate on diagonalizing the left-hand sides.

As will be seen shortly, it is useful to define "level" for tensors: lower $+$/upper $-$ indices contribute $+1$ to the level; lower $-$/upper $+$ indices contribute -1 to level; and the upper A/lower A indices contribute zero to the level. We shall see that the field equations should be solved in ascending order of their levels. The following is the detailed prescription of the diagonalization procedure. Let us use the shorthand notation $(E.E.)_{\mu\nu}$ for the lower $(\mu\nu)$ component of the Einstein equation, and $(M.E.)^{\mu_1\mu_2\mu_3}$ for the upper $(\mu_1\mu_2\mu_3)$ component of the Maxwell equation.

- at level -2

The only field equation at this level is $(E.E.)_{--}$, which reads, upon using the expressions of the various tensors $\nabla^\sigma \nabla_\sigma \bar{h}_{\mu\nu}, K_{\mu\nu}, Q_{\mu\nu}$... etc., that we have given above

$$-\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu \bar{h}_{--} + Q_{--} = \kappa_{11}^2 [T_{--}]_S \tag{63}$$

This equation can be immediately solved for $\bar{h}_{--}$ after specifying the source term and the gauge choice term Q_{--}.

- at level -1

We have $(E.E.)_{-A}$, which reads

$$-\frac{1}{2} \left[g^{\mu\nu} \partial_\mu \partial_\nu \bar{h}_{-A} - (\partial_A g_{++})(\partial_- \bar{h}_{--}) \right] + Q_{-A} = \kappa_{11}^2 [T_{-A}]_S \tag{64}$$

which can now be solved for $\bar{h}_{-A}$, using the $\bar{h}_{--}$ found previously. Also at this level is $(M.E.)^{+AB}$, which reads,

$$\begin{aligned}
& g^{\mu\nu} \partial_\mu \partial_\nu a_{-ij} - \partial_- (\nabla^\mu a_{\mu ij}) + \partial_i (\nabla^\mu a_{\mu - j}) - \partial_j (\nabla^\mu a_{\mu - i}) + \mu \epsilon_{ijk} (\partial_- \bar{h}_{-k} - \partial_k \bar{h}_{--}) = J^{+ij} \\
& g^{\mu\nu} \partial_\mu \partial_\nu a_{-ib} - \partial_- (\nabla^\mu a_{\mu ib}) + \partial_i (\nabla^\mu a_{\mu - b}) - \partial_b (\nabla^\mu a_{\mu - i}) = J^{+ib} \\
& g^{\mu\nu} \partial_\mu \partial_\nu a_{-bc} - \partial_- (\nabla^\mu a_{\mu bc}) + \partial_b (\nabla^\mu a_{\mu - c}) - \partial_c (\nabla^\mu a_{\mu - b}) = J^{+bc}
\end{aligned} \tag{65}$$

from which we can find a_{-AB}, upon specifying the gauge choice $\nabla^\mu a_{\mu\rho\lambda}$ for the three-form and using the $\bar{h}_{-A}$ and $\bar{h}_{--}$ found previously.

- at level 0

At this level we have $(E.E.)_{+-}$, $(M.E.)^{+-A}$, $(E.E.)_{AB}$, and $(M.E.)^{ABE}$.

$(E.E.)_{+-}$ is of the form

$$-\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu \bar{h}_{+-} = \text{known terms} \tag{66}$$

(From now on, we will not bother writing down the detailed equations; "known terms" refers to the gauge choice terms $Q_{\mu\nu}$, $\nabla^\mu a_{\mu\rho\lambda}$, source terms, and terms containing previously found $\bar{h}_{\mu\nu}$'s and $a_{\mu\nu\rho}$'s, one can write those down by looking up the expressions given earlier for the various tensors.) Hence solving it we get $\bar{h}_{+-}$. Solving $(M.E.)^{+-A}$ gives a_{-+A}.

$(E.E.)_{AB}$ and $(M.E.)^{ABE}$ are coupled, so a little more work is needed. The following are the details. First notice that the only unknown in $(M.E.)^{bce}$ is a_{bce}, hence solving this equation we find a_{bce} ($(M.E.)^{bce}$ contains the usual term $g^{\mu\nu} \partial_\mu \partial_\nu a_{bce}$ and also a term of the form $\partial_- a_{dfg}$ which comes from the $F \wedge F$ in the Maxwell equation, hence it is not quite a Laplace equation. But, that being said, one shouldn't have any difficulty solving it.)

$(M.E.)^{ibc}$ is of the form $g^{\mu\nu} \partial_\mu \partial_\nu a_{ibc} =$ known terms, solving which gives a_{ibc}. $(M.E.)^{ijb}$ and $(E.E.)_{kb}$ are coupled in the following manner

$$\begin{aligned}
& g^{\mu\nu} \partial_\mu \partial_\nu a_{ijb} + \mu \epsilon_{ijk} \partial_- \bar{h}_{kb} = \text{known terms} \\
& -\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu \bar{h}_{kb} + \frac{1}{4} \mu \epsilon_{klm} \partial_- a_{lmb} = \text{known terms}
\end{aligned} \tag{67}$$

Decoupling these two equations is quite easy. Let us take a_{12b} and $\bar{h}_{3b}$ as the representative case. One sees that these two equations can be recombined to give

$$\begin{aligned}(g^{\mu\nu}\partial_\mu\partial_\nu + i\mu\partial_-)(\bar{h}_{3b} + ia_{12b}) &= \text{known terms} \\ (g^{\mu\nu}\partial_\mu\partial_\nu - i\mu\partial_-)(\bar{h}_{3b} - ia_{12b}) &= \text{known terms}\end{aligned} \tag{68}$$

Solving these equations gives $(\bar{h}_{3b} + ia_{12b})$ and $(\bar{h}_{3b} - ia_{12b})$, and in turn $\bar{h}_{3b}$ and a_{12b}.

$(M.E.)^{ijk}$ is coupled to $(E.E.)_{ij}$ and $(E.E.)_{bd}$ through the quantity $H \equiv \frac{2}{3}\sum_{i=1}^{3}\bar{h}_{ii} - \frac{1}{3}\sum_{a=4}^{9}\bar{h}_{aa}$ in the following manner

$$\begin{aligned}g^{\mu\nu}\partial_\mu\partial_\nu a_{123} + \mu\partial_- H &= \text{known terms} \\ -\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{ij} + \frac{1}{2}\mu\delta_{ij}\partial_- a_{123} &= \text{known terms} \\ -\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{bd} - \frac{1}{2}\mu\delta_{bd}\partial_- a_{123} &= \text{known terms}\end{aligned} \tag{69}$$

Combining the last two equations gives

$$-g^{\mu\nu}\partial_\mu\partial_\nu H + 4\mu\partial_- a_{123} = \text{known terms} \tag{70}$$

Recombining this with first equation, we get

$$\begin{aligned}(g^{\mu\nu}\partial_\mu\partial_\nu + 2i\mu\partial_-)(H + 2ia_{123}) &= \text{known terms} \\ (g^{\mu\nu}\partial_\mu\partial_\nu - 2i\mu\partial_-)(H - 2ia_{123}) &= \text{known terms}\end{aligned} \tag{71}$$

solving which individually gives H and a_{123}. Using the resulting expression for a_{123} one can then find $\bar{h}_{ij}$ and $\bar{h}_{bd}$. Thus we are done with $(E.E.)_{AB}$ and $(M.E.)^{ABE}$.

- at level 1

$(M.E.)^{-AB}$ is of the form $g^{\mu\nu}\partial_\mu\partial_\nu a_{+AB} =$ known terms, solving which gives a_{+AB}.
$(E.E.)_{+A}$ is of the form $-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+A} =$ known terms, solving which gives $\bar{h}_{+A}$.

- at level 2

$(E.E.)_{++}$ is of the form $-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{++} =$ known terms, solving which gives $\bar{h}_{++}$.

Thus we have diagonalized the whole set of Einstein equations and Maxwell equations.

4.2 Application to a Spherical Membrane Source using the Near-Membrane Expansion

Now let us apply the general formalism of the previous subsection to the case of interest, with the source being a spherical membrane sitting at the origin of the transverse directions, i.e. having $(X^1)^2+(X^2)^2+(X^3)^2 = r_0^2$, $X^4 = 0, ..., X^9 = 0$, and $X^+ = t, X^- = 0$. The gauge

choice we shall take is: $q_\alpha = 0$ (hence all the $Q_{\alpha\beta}$'s vanish); and $\nabla^\mu a_{\mu\rho\lambda} = 0$. The nonzero components of the stress tensor and three-form current for this source are given by

$$
\begin{aligned}
&[T_{--}]_s = T\delta(x^-)\delta(r-r_0)\delta(x^4)...\delta(x^9)\left(\frac{\mu r_0}{3}\right)^{-1} \\
&[T_{+-}]_s = -\left(\frac{\mu r_0}{3}\right)^2[T_{--}]_s, \quad [T_{ij}]_s = \left(\frac{\mu r_0}{3}\right)^2\left(\frac{x^i x^j}{r_0^2} - \delta^{ij}\right)[T_{--}]_s, \quad [T_{++}]_s = \left(\frac{\mu r_0}{3}\right)^4[T_{--}]_s
\end{aligned} \tag{72}
$$

and

$$
J^{+ij} = \kappa_{11}^2(-2)\left(\frac{\mu r_0}{3}\right)\epsilon_{ijk}\frac{x^k}{r_0}[T_{--}]_s \tag{73}
$$

where $r \equiv \sqrt{x^i x^i}$.

Now let us explain what we mean by "near-membrane expansion". Define $w \equiv r - r_0$, $z \equiv \sqrt{x^a x^a}$, and $\xi \equiv \sqrt{w^2+z^2}$, which parameterize the distance away from the source membrane. We shall assume that w, z, ξ are of the same order of magnitude. The near-membrane expansion is an expansion in ξ/r_0. When one sits very close to membrane, one just sees a flat membrane, which is the zeroth order of the expansion. As one moves away from the membrane, one begins to feel the curvature of membrane, which gives the higher order corrections in the expansion. One should also note that the zeroth order of this expansion is just the flat space limit: $\mu \to 0$, $r_0 \to \infty$, with μr_0 kept finite.

It is instructive to see how the zeroth order works. At this order, in the Einstein equations and Maxwell equations, the effective source terms (which are of the forms $(\partial g_{++})\bar{h}_{\mu\nu}$, etc.) arising from the various tensors $K_{\mu\nu}$, $N_{\mu\nu}$, $L_{\mu\nu}$, $Z^{\mu\nu\rho}$ etc. are less singular then the delta-functional sources $[T_{\mu\nu}]_s$ and $J^{\mu\nu\rho}$, and can thus be thrown away. Then the resulting equations are trivially decoupled. Also, at this order, we can treat the x^i's in $[T_{\mu\nu}]_s$ and $J^{\mu\nu\rho}$ as constant vectors. One finds (using the subscript 0 to denote "zeroth order")

$$
\begin{aligned}
&[\bar{h}_{-A}]_0 = 0, \quad [a_{-ij}]_0 = \left(\frac{\mu r_0}{3}\right)\epsilon_{ijk}\frac{x^k}{r_0}[\bar{h}_{--}]_0, \quad [a_{-ib}]_0 = 0, \quad [a_{-bc}]_0 = 0 \\
&[\bar{h}_{+-}]_0 = -\left(\frac{\mu r_0}{3}\right)^2[\bar{h}_{--}]_0, \quad [a_{-+A}]_0 = 0, \quad [\bar{h}_{ij}]_0 = \left(\frac{\mu r_0}{3}\right)^2\left(\frac{x^i x^j}{r_0^2} - \delta^{ij}\right)[\bar{h}_{--}]_0 \\
&[\bar{h}_{ib}]_0 = 0, \quad [\bar{h}_{bc}]_0 = 0, \quad [a_{ABE}]_0 = 0 \\
&[a_{+ij}]_0 = -\left(\frac{\mu r_0}{3}\right)^3\epsilon_{ijk}\frac{x^k}{r_0}[\bar{h}_{--}]_0, \quad [a_{+ib}]_0 = 0, \quad [a_{+bc}]_0 = 0, \quad [\bar{h}_{+A}]_0 = 0 \\
&[\bar{h}_{++}]_0 = \left(\frac{\mu r_0}{3}\right)^4[\bar{h}_{--}]_0
\end{aligned} \tag{74}
$$

where $[\bar{h}_{--}]_0$ satisfies

$$
\left[-\left(\frac{\mu r_0}{3}\right)^2 k_-^2 + \frac{\partial^2}{(\partial w)^2} + \frac{\partial^2}{(\partial x^4)^2} + ...\frac{\partial^2}{(\partial x^9)^2}\right][\bar{h}_{--}]_0 = -\frac{1}{\pi R}\left(\frac{\mu r_0}{3}\right)^{-1}\kappa_{11}^2 T\delta(w)\delta(x^4)...\delta(x^9) \tag{75}
$$

(where we have multiplied the right hand side of the equation by $\frac{1}{2\pi R}$ due to the Fourier transform along the x^- direction), and is given by

$$[\bar{h}_{--}]_0 = \Delta \frac{\exp\left(-\frac{\mu r_0}{3}k_-\xi\right)}{\xi^5}\left[3 + 3\left(\frac{\mu r_0}{3}k_-\xi\right) + \left(\frac{\mu r_0}{3}k_-\xi\right)^2\right] \tag{76}$$

with $\Delta \equiv \frac{\kappa_{11}^2 T}{16\pi^4 R\left(\frac{\mu r_0}{3}\right)}$.

Plugging the above zeroth order solution of the field equations into the light-cone Lagrangian $\delta\mathcal{L}_{lc}$ given in eqn (27), for a spherical probe membrane with radius r_0', sitting at rest in the $1, 2, 3$ directions, and moving about in the 4 through 9 directions: $(X^1)^2 + (X^2)^2 + (X^3)^2 = r_0'^2$, $X^4(t), ..., X^9(t)$, and $X^+ = t, X^- = 0$, one finds (using the facts that $T = \frac{\mu\Pi_-}{3r_0'\sin\theta}$, and we can set $r_0' = r_0$ because we are looking at the zeroth order)

$$\delta\mathcal{L}_{lc} = \frac{1}{8}\Pi_-[\bar{h}_{--}]_0(\dot{X}^a\dot{X}^a)^2 \tag{77}$$

It is worth noting that, keeping only the leading order term in k_- in $[\bar{h}_{--}]_0$, eqn (77) becomes the v^4 Lagrangian for the case of longitudinal momentum transfer between two membranes in the flat space, given in [16].

Now let us go on to consider higher orders in the near-membrane expansion. Since in this paper we do not consider longitudinal momentum transfer, we shall set $k_- = 0$ (which makes many fields equations decouple).

Denote

$$\begin{aligned} \Box &\equiv \partial_A\partial_A \\ &\quad\text{(when acting on functions of } (w, z)) \\ &= \Box_0 + \delta\Box \end{aligned} \tag{78}$$

with $\Box_0 \equiv \frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial z^2} + \frac{5}{z}\frac{\partial}{\partial z} = \partial_w^2 + \partial_a\partial_a$ being the zeroth order Laplace operator, and $\delta\Box \equiv \frac{2}{r_0+w}\frac{\partial}{\partial w}$ being curvature correction to it.

At level -2

$(E.E.)_{--}$

$$\Box\bar{h}_{--} = -\frac{1}{\pi R}\kappa_{11}^2 T\delta(w)\delta(x^4)...\delta(x^9)\left(\frac{\mu r_0}{3}\right)^{-1} \tag{79}$$

Let

$$\bar{h}_{--} = [\bar{h}_{--}]_0 + \delta\bar{h}_{--} \tag{80}$$

with $[\bar{h}_{--}]_0 \equiv \Delta\frac{3}{\xi^5}$, which satisfies

$$\Box_0[\bar{h}_{--}]_0 = -\frac{1}{\pi R}\kappa_{11}^2 T\delta(w)\delta(x^4)...\delta(x^9)\left(\frac{\mu r_0}{3}\right)^{-1} \tag{81}$$

Then $(E.E.)_{--}$ becomes

$$\Box_0 \delta \bar{h}_{--} + \delta\Box[\bar{h}_{--}]_0 + \delta\Box\delta\bar{h}_{--} = 0 \tag{82}$$

Now let's look at the order of magnitude of each term in the above equation. Notice that $\Box_0 \sim \frac{1}{\xi^2}$, $\delta\Box \sim \frac{1}{r_0 \xi}$. The second term is thus $\sim \Delta\frac{1}{r_0\xi^6}$, which tells us $\delta\bar{h}_{--} \sim \Delta\frac{1}{r_0\xi^4}$. Solving the equation iteratively we find

$$\delta\bar{h}_{--} = [\bar{h}_{--}]_0 \left(-\frac{w}{r_0} + \frac{w^2}{r_0^2} - \frac{w^3}{r_0^3} + \frac{w^4}{r_0^4}\right) \tag{83}$$

and thus

$$\bar{h}_{--} = 3\Delta\frac{1}{\xi^5}\left[\frac{r_0}{r_0 + w} + O\left(\frac{\xi^5}{r_0^5}\right)\right] \tag{84}$$

We did not compute the $O\left(\frac{\xi^5}{r_0^5}\right)$ terms, because we are only interested in the part of the solution that is singular as $\xi \to 0$. Solving the other field equations is similar, so we just present the results below, omitting the $O\left(\frac{\xi^5}{r_0^5}\right)$ symbol.

At level -1

$$\begin{aligned}
&\bar{h}_{-A} = 0 \\
&a_{-bc} = 0 \\
&a_{-ib} = 0 \\
&a_{-ij} = \epsilon_{ijk}x^k\mu\Delta\frac{1}{\xi^5}\left[1 - \frac{1}{2}\frac{w}{r_0} + \frac{1}{6}\frac{w^2+z^2}{r_0^2} - \frac{1}{2}\frac{wz^2}{r_0^3} + \frac{w^2z^2}{r_0^4}\right]
\end{aligned} \tag{85}$$

At level 0

$$\begin{aligned}
&\bar{h}_{+-} = -\left(\frac{\mu r_0}{3}\right)^2\frac{3\Delta}{\xi^5}\left[1 + \left(\frac{5}{4}\frac{w^2}{r_0^2} + \frac{7}{8}\frac{z^2}{r_0^2}\right)\frac{r_0}{r_0+w}\right] \\
&a_{-+A} = 0 \\
&a_{ABD} = 0
\end{aligned} \tag{86}$$

$$\bar{h}_{kb} = x^k x^b \mu^2\Delta\frac{1}{\xi^5}\left(-\frac{1}{2}\right)\left[1 - \frac{5}{4}\frac{w}{r_0} + \frac{17}{12}\frac{w^2}{r_0^2} - \frac{1}{12}\frac{z^2}{r_0^2} - \frac{3}{2}\frac{w^3}{r_0^3} + \frac{1}{4}\frac{wz^2}{r_0^3} + \frac{3}{2}\frac{w^4}{r_0^4} - \frac{1}{2}\frac{w^2z^2}{r_0^4}\right] \tag{87}$$

$$\begin{aligned}
\bar{h}_{ij} = \quad & \frac{x^i x^j}{r^2}\left(\frac{\mu r_0}{3}\right)^2\Delta\frac{3}{\xi^5}\left[1 - \frac{w}{r_0} - \frac{z^2}{r_0^2} + \frac{w^3}{r_0^3} + 2\frac{wz^2}{r_0^3} - \frac{w^4}{r_0^4} - \frac{w^2z^2}{r_0^4} + \frac{z^4}{r_0^4}\right] \\
& -\delta^{ij}\left(\frac{\mu r_0}{3}\right)^2\Delta\frac{3}{\xi^5}\left[1 + \frac{1}{2}\frac{w}{r_0} + \frac{7}{12}\frac{w^2}{r_0^2} - \frac{1}{24}\frac{z^2}{r_0^2} - \frac{1}{4}\frac{w^3}{r_0^3} + \frac{3}{8}\frac{wz^2}{r_0^3} + \frac{1}{4}\frac{w^4}{r_0^4} - \frac{1}{24}\frac{w^2z^2}{r_0^4} + \frac{1}{3}\frac{z^4}{r_0^4}\right]
\end{aligned} \tag{88}$$

$$\bar{h}_{bd} = \delta_{bd}(\mu r_0)^2 \Delta \frac{1}{\xi^5}\left[\frac{1}{2}\frac{w}{r_0} - \frac{7}{18}\frac{w^2}{r_0^2} - \frac{19}{72}\frac{z^2}{r_0^2} + \frac{7}{18}\frac{w^3}{r_0^3} + \frac{19}{72}\frac{wz^2}{r_0^3} - \frac{7}{18}\frac{w^4}{r_0^4} - \frac{19}{72}\frac{w^2z^2}{r_0^4}\right] \tag{89}$$

At level +1

$$a_{+bc} = 0$$
$$a_{+ib} = 0$$
$$a_{+ij} = -\epsilon_{ijk}x^k\left(\frac{\mu r_0}{3}\right)^2 \mu\Delta\frac{1}{\xi^5}\left[1 + 2\frac{w}{r_0} - \frac{w^2}{r_0^2} - \frac{5}{4}\frac{z^2}{r_0^2} + \frac{3}{2}\frac{w^3}{r_0^3} + \frac{7}{4}\frac{wz^2}{r_0^3} - \frac{3}{2}\frac{w^4}{r_0^4} - \frac{5}{4}\frac{w^2z^2}{r_0^4} + \frac{1}{2}\frac{z^4}{r_0^4}\right] \tag{90}$$

$$\bar{h}_{+A} = 0 \tag{91}$$

At level +2

$$\bar{h}_{++} = \left(\frac{\mu r_0}{3}\right)^4 \Delta\frac{3}{\xi^5}\left[1 + \frac{5}{2}\frac{w}{r_0} + \frac{31}{12}\frac{w^2}{r_0^2} - \frac{1}{24}\frac{z^2}{r_0^2} + \frac{17}{12}\frac{w^3}{r_0^3} - \frac{1}{12}\frac{wz^2}{r_0^3} + \frac{1}{3}\frac{w^4}{r_0^4} + \frac{23}{24}\frac{w^2z^2}{r_0^4} + \frac{17}{32}\frac{z^4}{r_0^4}\right] \tag{92}$$

(A note in passing: as it turns out, the choice of convention for $\tilde{\eta}$ in the Maxwell equation (54) does not matter, since it drops out when solving the field equations for our specific source.)

Again, let the probe membrane have a radius $r_0' = r_0 + w$, with the trajectory $(X^1)^2 + (X^2)^2 + (X^3)^2 = r_0'^2$, $X^4(t), ..., X^9(t)$, and $X^+ = t, X^- = 0$. We shall take $v \equiv \sqrt{\dot{X}^a\dot{X}^a}$ to be of order μz (recall that the supersymmetric circular orbit has $v = \frac{1}{6}\mu z$; so here we are considering generically nonsupersymmetric orbits that can be regarded as deformations of the supersymmetric circular one). Plugging in the supergravity solution given above into eqn (27), using $T = \frac{\mu\Pi_-}{3r_0'\sin\theta}$, and in the end keeping only the part of $\delta\mathcal{L}_{lc}$ that is singular as $\xi \to 0$, we find the probe's $\mathcal{L}_{lc}$ to be

$$\mathcal{L}_{lc} = (\mathcal{L}_{lc})_{pp} + \delta\mathcal{L}_{lc} \tag{93}$$

with $(\mathcal{L}_{lc})_{pp}$ being the action in the unperturbed pp-wave background, and

$$\delta\mathcal{L}_{lc} = \Pi_-\Delta\frac{\mu^4(4w^2z^2 + 7z^4) - 72\mu^2v^2(2w^2 + 5z^2) + 3888v^4}{10368\xi^5} \tag{94}$$

Notice that the above $\delta\mathcal{L}_{lc}$'s singular behavior as $\xi \to 0$ is homogeneous: $\sim \frac{1}{\xi}$ (since v is of order μz). The expression in the numerator: $\mu^4(4w^2z^2 + 7z^4) - 72\mu^2v^2(2w^2 + 5z^2) + 3888v^4$ nicely factorizes into $(36v^2 - z^2\mu^2)[108v^2 - (4w^2 + 7z^2)\mu^2]$, which shows that for the special case of the supersymmetric circular orbit $v = \frac{1}{6}\mu z$ considered in [10], δL_{lc} vanishes as expected.

To be more precise, so far we have been talking about Lagrangian density. Since the membrane worldvolume is taken to be a unit sphere, the Lagragian is given by

$$\begin{aligned}
\delta L_{lc} = \int d\theta d\phi \delta\mathcal{L}_{lc} &= \frac{\kappa_{11}^2 T^2}{4\pi^3 R\mu^2} \frac{(36v^2 - z^2\mu^2)[108v^2 - (4w^2 + 7z^2)\mu^2]}{1152\xi^5} \\
&= \frac{\alpha}{\mu^2} \frac{(36v^2 - z^2\mu^2)[108v^2 - (4w^2 + 7z^2)\mu^2]}{1152\xi^5}
\end{aligned} \tag{95}$$

where to get the first line we used $T = \frac{\mu\Pi_-}{3r_0' \sin\theta}$ to eliminate Π_- in terms of T and r_0', and set $r_0' \approx r_0$ in the end to remove higher $\frac{w}{r_0}$ order curvature correction. To get the second line we used the expressions for κ_{11}^2, T, and α in terms of M and R given at the end of Subsection 2.1.

Here we would like to make a brief comparison of the above membrane result with the graviton result given in [2].

First of all, the membrane result contains the variable w (the difference in radius between the probe membrane and the source membrane), which has no counterpart in the graviton case. Secondly, in terms of the x^1 through x^3 directions, the two membranes are sitting at rest at the origin; this corresponds to setting $x^i = 0$ and $v_i = 0$ in the graviton case.

If we set $w = 0$ in eqn (95), i.e., consider two membranes of the same size, then $\xi = z$, and

$$(\delta L_{lc})_{\text{membrane}} = \frac{\alpha}{1152\mu^2 z^5}(36v^2 - z^2\mu^2)(108v^2 - 7z^2\mu^2) \tag{96}$$

while for the gravitons, upon setting $N_p = N_s = N$, $x^i = 0$, and $v_i = 0$ in eqn (19) of [2], we have

$$(\delta L_{lc})_{\text{graviton}} = \frac{\alpha^3 N^2}{5736 z^7}(36v^2 - z^2\mu^2)(140v^2 - 7z^2\mu^2) \tag{97}$$

When comparing the above two expressions, note the difference between the numerators: $(108v^2 - 7z^2\mu^2)$ for membrane, and $(140v^2 - 7z^2\mu^2)$ for graviton. Also notice their different power law dependence on z: $\frac{1}{z^5}$ for membrane and $\frac{1}{z^7}$ for graviton, which cannot be undone by integrating over the membrane (r_0, the radius of the membrane, has nothing to do with z, the separation of the membranes in the X^4 to X^9 directions).

5 Matrix Theory Computation - The Membrane Limit

Shin and Yoshida have previously calculated the one-loop effective action for membrane fuzzy spheres extended in the first three directions and having periodic motion in a sub-plane of the remaining six transverse directions. In reference [10] they considered the case

of supersymmetric circular motion for an arbitrary radius and angular frequency $\frac{\mu}{6}$ (this orbit preserves eight supersymmetries and was first found by [1]). Here we generalize that analysis to orbits which are not required to satisfy the classical equations of motion and are in general nonsupersymmetric. The procedures are: expanding the action to quadratic order in fluctuations, writing the fields in terms of the matrix spherical harmonics introduced in [17] [2] , diagonalizing the mass matrices of the bosons, fermions, and ghosts, and finally summing up the masses to get the one-loop effective action. In doing so, we shall adopt the notations of [10]. We shall consider the background

$$B^I = \begin{pmatrix} B^I_{(1)} & 0 \\ 0 & B^I_{(2)} \end{pmatrix} \tag{98}$$

where

$$\begin{aligned} B^i_{(1)} &= \frac{\mu}{3} J^i_{(1)N_1\times N_1} & \qquad B^a_{(1)} &= 0 \cdot \ 1_{N_1\times N_1} \\ B^i_{(2)} &= \frac{\mu}{3} J^i_{(2)N_2\times N_2} + x^i(t) 1_{N_2\times N_2} & \qquad B^a_{(2)} &= x^a(t) 1_{N_2\times N_2} \end{aligned} \tag{99}$$

with the J^i's being $su(2)$ generators. The above background has the interpretation of one spherical membrane (labeled by the subscript (1)) sitting at the origin, and the other spherical membrane (labeled by the subscript (2)) moving along the arbitrary orbit given by $\{x^i(t), x^a(t)\}$.

The fluctuations of the bosonic fields are given by

$$\begin{aligned} A &= \begin{pmatrix} Z^0_{(1)} & \Phi^0 \\ (\Phi^0)^\dagger & Z^0_{(2)} \end{pmatrix} \\ Y^I &= \begin{pmatrix} Z^I_{(1)} & \Phi^I \\ (\Phi^I)^\dagger & Z^I_{(2)} \end{pmatrix} \end{aligned} \tag{100}$$

As it turns out, the part of the bosonic action containing the diagonal fluctuations Z^0, Z^I does not contain any new terms in addition to those given in [10]. So we do not write it out here. (It shall be the same situation for the fermionic and ghost parts of the action; it is the off-diagonal fluctuations that give the one-loop interaction potential between the two

[2]For the computation below, one only needs the transformation of the matrix spherical harmonics under $SU(2)$, however, for detailed construction of the matrix spherical harmonics, see, e.g., Appendix A of [18].

membranes.) The action for the off-diagonal fluctuations is

$$
\begin{aligned}
S_{OD} = \int dt\, Tr\, [& -|\dot{\Phi}^0|^2 + x^2|\Phi^0|^2 + \left(\frac{\mu}{3}\right)^2 |J^i \circ \Phi^0|^2 - 2\left(\frac{\mu}{3}\right) x^i Re((J^i \circ \Phi^0)(\Phi^0)^\dagger) \\
& +|\dot{\Phi}^i|^2 - x^2|\Phi^i|^2 - \left(\frac{\mu}{3}\right)^2 |\Phi^i + i\epsilon^{ijk} J^j \circ \Phi^k|^2 + \left(\frac{\mu}{3}\right)^2 |J^i \circ \Phi^i|^2 \\
& +2\left(\frac{\mu}{3}\right) x^i Re((J^i \circ \Phi^0)(\Phi^0)^\dagger) - 2i\dot{x^i}((\Phi^0)^\dagger \Phi^i - (\Phi^i)^\dagger \Phi^0) \\
& +|\dot{\Phi}^a|^2 - \left(x^2 + \frac{\mu^2}{6^2}\right)|\Phi^a|^2 - \left(\frac{\mu}{3}\right)^2 |J^i \circ \Phi^a|^2 \\
& + 2\left(\frac{\mu}{3}\right) x^i Re((J^i \circ \Phi^a)(\Phi^a)^\dagger) - 2i\dot{x^a}((\Phi^0)^\dagger \Phi^a - (\Phi^a)^\dagger \Phi^0)\Big]
\end{aligned} \tag{101}
$$

where $x^2 \equiv x^i x^i + x^a x^a$, and dots means time-derivatives.

We specialize to the case where, $x^i = 0, x^8 = b, x^9 = vt$, with $(x^8)^2 + (x^9)^2$ denoted by z^2. The effective potential will be computed by summing over the mass of the fermionic and ghost fluctuations and then subtracting the mass of the bosonic fluctuations. This method, which we will refer to as the sum over mass method, is the same as the one used in [2]. Although the above trajectory has the form of a straight line with constant velocity in the (x^8, x^9) plane, the final expression of V_{eff} in terms of $z \equiv \sqrt{(x^a)^2}$ and $v \equiv \sqrt{(\dot{x^a})^2}$ should suffice for the purpose of comparing with supergravity for arbitrary orbits $x^a(t)$. One may ask whether the sum over mass formula is valid when the masses of the fluctuations are time-dependent (one origin for such a time-dependence is the acceleration of the trajectory). In Section 4.2 and Appendix A of [2], it was carefully shown that, in the case of two-graviton interaction in the pp-wave background, the sum over mass formula was sufficient in computing the terms that could occur on the supergravity side. The time-dependence in the masses of the fluctuations will give terms of the form of matrix theory corrections to supergravity (i.e., terms that dominate at extreme short distances and cannot be observed in supergravity), which does not concern us since we are only interested in a comparison with supergravity. Here we expect a similar argument to hold in the case of two-membrane interaction. The rather non-trivial agreement with supergravity presented at the end of this section and also the agreement with the work of Shin and Yoshida [11] in section 6.2 confirm the validity of the sum over mass method.

Expand the fields in terms of matrix spherical harmonics

$$
\Phi^{0,I} = \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \sum_{m=-j}^{j} \phi^{0,I}_{jm} Y^{N_1\times N_2}_{jm}. \tag{102}
$$

For our choice of background the masses of modes in the $i = 1,2,3$ and $a = 4,5,6,7,8$

directions are the same as those in [10]. For the gauge field and $a = 9$ direction we have

$$S = \int dt \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \Big[-|\dot{\phi}^0_{jm}|^2 + \left(z^2 + \left(\frac{\mu}{3}\right)^2 j(j+1) \right) |\phi^0_{jm}|^2$$
$$+|\dot{\phi}^9_{jm}|^2 - \left(z^2 + \left(\frac{\mu}{6}\right)^2 + \left(\frac{\mu}{3}\right)^2 j(j+1) \right) |\phi^0_{jm}|^2$$
$$-2iv \left(\left(\phi^0_{jm}\right)^* \phi^9_{jm} - \left(\phi^9_{jm}\right)^* \phi^0_{jm} \right) \Big] \tag{103}$$

where there is an implicit sum over $-j \leq m \leq j$. It is straightforward to diagonalize the mass matrix for these modes. Combining all contributions from bosonic fluctuations we get an bosonic effective potential[3] given by,

$$\begin{aligned} V^B_{eff} = & - \sum_{j=\frac{1}{2}|N_1-N_2|-1}^{\frac{1}{2}(N_1+N_2)-2} (2j+1)\sqrt{z^2 + \left(\frac{\mu}{3}\right)^2 (j+1)^2} \\ & - \sum_{j=\frac{1}{2}|N_1-N_2|+1}^{\frac{1}{2}(N_1+N_2)} (2j+1)\sqrt{z^2 + \left(\frac{\mu}{3}\right)^2 j^2} \\ & - \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} (2j+1)\sqrt{z^2 + \left(\frac{\mu}{3}\right)^2 j(j+1)} \\ & - \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} 5(2j+1)\sqrt{z^2 + \left(\frac{\mu}{3}\right)^2 (j+\frac{1}{2})^2} \\ & - \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} (2j+1) \left(\sqrt{z^2 + \left(\frac{\mu}{3}\right)^2 (j(j+1)+\frac{1}{8}) + \frac{1}{2}\sqrt{\left(\frac{\mu}{6}\right)^4 + 16v^2}} \right. \\ & \left. + \sqrt{z^2 + \left(\frac{\mu}{3}\right)^2 (j(j+1)+\frac{1}{8}) - \frac{1}{2}\sqrt{\left(\frac{\mu}{6}\right)^4 + 16v^2}} \right) \end{aligned} \tag{104}$$

Now for the fermions we start with the action given in [10]

$$L_F = Tr(i\Psi^\dagger\dot{\Psi} - \Psi^\dagger\gamma^I[\Psi, B^I] - i\frac{\mu}{4}\Psi^\dagger\gamma^{123}\Psi) \tag{105}$$

with

$$\Psi = \begin{pmatrix} \Psi_{(1)} & \chi \\ \chi^\dagger & \Psi_{(2)} \end{pmatrix}. \tag{106}$$

[3] Note that our convention differs from that of [10] by an overall minus sign. See section 2.

Again the action for the diagonal fluctuations has no new terms, and for the off-diagonal part we decompose the $SO(9)$ spinor χ using the subgroup $SO(3)\times SO(6)\sim SU(2)\times SU(4)$ preserved by PP-wave, $\mathbf{16}\rightarrow(\mathbf{2},\mathbf{4})+(\bar{\mathbf{2}},\bar{\mathbf{4}})$

$$\chi=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi_{A\alpha}\\ \hat{\chi}^{A_\alpha}\end{pmatrix}. \tag{107}$$

Substituting this into L_F,

$$\begin{aligned}L_F=Tr\Big[&i(\chi^\dagger)^{A\alpha}\dot{\chi}_{A\alpha}-\frac{1}{4}(\chi^\dagger)^{A\alpha}\chi_{A\alpha}-\frac{\mu}{3}(\chi^\dagger)^{A\alpha}(\sigma^i)_\alpha^\beta J^i\circ\chi_{A\beta}\\ &+i(\hat{\chi}^\dagger)^{A\alpha}\dot{\hat{\chi}}_{A\alpha}+\frac{1}{4}(\hat{\chi}^\dagger)^{A\alpha}\hat{\chi}_{A\alpha}+\frac{\mu}{3}(\hat{\chi}^\dagger)^{A\alpha}(\sigma^i)_\alpha^\beta J^i\circ\hat{\chi}_{A\beta}\\ &+x^i(-(\chi^\dagger)^{A\alpha}(\sigma^i)_\alpha^\beta\chi_{A\beta}+(\hat{\chi}^\dagger)^{A\alpha}(\sigma^i)_\alpha^\beta\hat{\chi}_{A\beta})+x^a((\chi^\dagger)^{A\alpha}\rho^a_{AB}\hat{\chi}^B_\alpha+(\hat{\chi}^\dagger)^\alpha_A((\rho^a)^{AB})^\dagger\chi_{B\alpha})\Big]\end{aligned} \tag{108}$$

with the σ^i's and ρ^a's being the gamma matrices for $SO(3)$ and $SO(6)$ respectively. We now substitute our specific background and expand in modes,

$$\begin{aligned}L_F\;=\;&\sum_{j=\frac{1}{2}|N_1-N_2|-1/2}^{\frac{1}{2}(N_1+N_2)-3/2}\Big[i\pi^\dagger_{jm}\dot{\pi}_{jm}+i\hat{\pi}^\dagger_{jm}\dot{\hat{\pi}}_{jm}-\frac{1}{3}\left(j+\frac{3}{4}\right)(\pi^\dagger_{jm}\pi_{jm}-\hat{\pi}^\dagger_{jm}\hat{\pi}_{jm})\\ &\qquad+x^a(\pi^\dagger_{jm}\rho^a\hat{\pi}_{jm}+\hat{\pi}^\dagger_{jm}(\rho^a)^\dagger\pi_{jm})\Big]\\ &+\sum_{j=\frac{1}{2}|N_1-N_2|+1/2}^{\frac{1}{2}(N_1+N_2)-1/2}\Big[i\eta^\dagger_{jm}\dot{\eta}_{jm}+i\hat{\eta}^\dagger_{jm}\dot{\hat{\eta}}_{jm}-\frac{1}{3}\left(j+\frac{1}{4}\right)(\eta^\dagger_{jm}\eta_{jm}-\hat{\eta}^\dagger_{jm}\hat{\eta}_{jm})\\ &\qquad+x^a(\eta^\dagger_{jm}\rho^a\hat{\eta}_{jm}+\hat{\eta}^\dagger_{jm}(\rho^a)^\dagger\eta_{jm})\Big]\end{aligned} \tag{109}$$

where again there is an implicit sum over m. This can be diagonalized and contributes to the effective action

$$\begin{aligned}V^F_{eff}=\;&2\sum_{j=\frac{1}{2}|N_1-N_2|-1/2}^{\frac{1}{2}(N_1+N_2)-3/2}(2j+1)(\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j+\frac{3}{4})^2+v}+\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j+\frac{3}{4})^2-v}\\ &+2\sum_{j=\frac{1}{2}|N_1-N_2|+1/2}^{\frac{1}{2}(N_1+N_2)-1/2}(2j+1)(\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j+\frac{1}{4})^2+v}+\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j+\frac{1}{4})^2-v}\end{aligned} \tag{110}$$

There is also the contribution from the ghosts, however, this has no new terms for our choice of background,

$$V^G_{eff}=2\sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1}(2j+1)\sqrt{z^2+\left(\frac{\mu}{3}\right)^2j(j+1)} \tag{111}$$

and the total effective action is the sum of the three pieces

$$V_{eff} = V_{eff}^B + V_{eff}^F + V_{eff}^G. \tag{112}$$

We introduce the variables N and u,

$$N_1 = N + 2u \qquad N_2 = N, \tag{113}$$

where u will be related to the difference in radii of the two spheres and from now on we will restore α using dimensional analysis. Define

$$\begin{aligned} \eta_\pm^2 &= z^2 \pm \frac{1}{2}\sqrt{\left(\frac{\alpha\mu}{6}\right)^4 + 16\alpha^2 v^2} \\ \nu_\pm^2 &= z^2 \pm \alpha v \end{aligned} \tag{114}$$

and assuming that $u \geq 1$ (rather than assuming $u \geq 0$ because the lower limit of the first summation in eqn (104) has to be non-negative) we can write the effective action so that j always starts from 0 and finishes at $N-1$.

$$\begin{aligned} V_{eff} = -\frac{1}{\alpha}\sum_{j=0}^{N-1}\Bigg\{ & (2j+2u-1)\left[z^2 + \left(\frac{\alpha\mu}{3}\right)^2 (j+u)^2\right]^{\frac{1}{2}} \\ & +(2j+2u+3)\left[z^2 + \left(\frac{\alpha\mu}{3}\right)^2 (j+u+1)^2\right]^{\frac{1}{2}} \\ & +(2j+2u+1)\left[z^2 + \left(\frac{\alpha\mu}{3}\right)^2 (j+u)(j+u+1)\right]^{\frac{1}{2}} \\ & +5(2j+2u+1)\left[z^2 + \left(\frac{\alpha\mu}{3}\right)^2 (j+u+1/2)^2\right]^{\frac{1}{2}} \\ & +(2j+2u+1)\Bigg(\left[\eta_+^2 + \left(\frac{\alpha\mu}{3}\right)^2 ((j+u)(j+u+1)+\frac{1}{8})\right]^{\frac{1}{2}} \\ & \qquad + \left[\eta_-^2 + \left(\frac{\alpha\mu}{3}\right)^2 ((j+u)(j+u+1)+\frac{1}{8})\right]^{\frac{1}{2}}\Bigg) \\ & -2(2j+2u)\left(\left[\nu_+^2 + \left(\frac{\alpha\mu}{3}\right)^2 (j+u+1/4)^2\right]^{\frac{1}{2}} + \left[\nu_-^2 + \left(\frac{\alpha\mu}{3}\right)^2 (j+u+1/4)^2\right]^{\frac{1}{2}}\right) \\ & -2(2j+2u+2)\left(\left[\nu_+^2 + \left(\frac{\alpha\mu}{3}\right)^2 (j+u+3/4)^2\right]^{\frac{1}{2}} + \left[\nu_-^2 + \left(\frac{\alpha\mu}{3}\right)^2 (j+u+3/4)^2\right]^{\frac{1}{2}}\right) \\ & -2(2j+2u+1)\left[z^2 + \left(\frac{\alpha\mu}{3}\right)^2 (j+u)(j+u+1)\right]^{\frac{1}{2}}\Bigg\} \end{aligned} \tag{115}$$

Let us write the above summation as

$$V_{eff} = \sum_{j=0}^{N-1} \mathcal{V}(j) \tag{116}$$

and use Euler-Maclaurin sum formula to convert it into integrals

$$\begin{aligned} V_{eff} = \int_0^N \mathcal{V}(j)dj - \frac{1}{2}[\mathcal{V}(N) + \mathcal{V}(0)] + \frac{1}{12}[\mathcal{V}'(N) - \mathcal{V}'(0)] \\ -\frac{1}{720}[\mathcal{V}'''(N) - \mathcal{V}'''(0)] + \cdots \end{aligned} \tag{117}$$

It is useful to first see what we are expecting from such an integral. From eqn(115), we see that a typical term of $\mathcal{V}(j)$ is roughly of the form:

$$\begin{aligned} \mathcal{V}(j) &= \frac{1}{\alpha} j\sqrt{z^2 + \alpha v + \alpha^2\mu^2 j^2} \\ &= \mu N^2 \frac{j}{N}\sqrt{(\frac{j}{N})^2 + \frac{1}{N^2}((\frac{z}{\alpha\mu})^2 + \frac{v}{\alpha\mu^2})} \end{aligned} \tag{118}$$

Plugging the above form of $\mathcal{V}$ into eqn (117) and defining new variables $\gamma = j/N$ and $\zeta = \sqrt{((\frac{z}{\alpha\mu})^2 + \frac{v}{\alpha\mu^2})}$, not keeping track of the exact coefficients, we have:

$$\begin{aligned} V_{eff} &= \mu N^3 \int_0^1 d\gamma\ \gamma\sqrt{\gamma^2 + (\frac{\zeta}{N})^2} + \mu N^2 \left(\sqrt{1 + (\frac{\zeta}{N})^2}\right) \\ &\quad + \mu N \left(\partial_\gamma \left(\gamma\sqrt{\gamma^2 + (\frac{\zeta}{N})^2}\right)\right) \Bigg|_0^1 + \frac{\mu}{N}\left(\partial_\gamma^3 \left(\gamma\sqrt{\gamma^2 + (\frac{\zeta}{N})^2}\right)\right) \Bigg|_0^1 + \cdots \\ &= \mu N^3 \left\{ F_0(\frac{\zeta}{N}) + \frac{1}{N}F_1(\frac{\zeta}{N}) + \frac{1}{N^2}F_2(\frac{\zeta}{N}) + \cdots \right\} \end{aligned} \tag{119}$$

where F_n are functions of $\frac{\zeta}{N} = \frac{1}{N}\sqrt{((\frac{z}{\alpha\mu})^2 + \frac{v}{\alpha\mu^2})}$ originating from the n-th derivative of γ (we shall see in the next paragraph why we use $\frac{\zeta}{N}$ as the argument for F_n). Note that each F_n is weighted by a factor $\frac{1}{N^n}$.

First we look at the term F_0. Assuming in an power expansion of $F_0(x)$ there exists a term x^3, after expanding in α it would contribute to V_{eff} a v^4 term of the form $\mu N^3(\frac{\zeta}{N})^3 \sim \frac{\alpha v^4}{\mu^2 z^5}$, which has the correct form of membrane interactions and has the correct order of N (see section 2). If in the power expansion of $F_0(x)$ there also exists a term x^1, then it would contribute to V_{eff} a v^4 term of the form $\mu N^3(\frac{\zeta}{N}) \sim \frac{N^2\alpha^3 v^4}{z^7}$. This is the same expression as graviton interactions. The details of the coefficients and whether such terms exist in the power expansion of course has to be seen by actually performing the integration, however as

we will see later, the integrals do produce terms of the correct interactions, in the both the membrane and the graviton limit.

Next we look at F_n for $n > 0$. The whole argument in the last paragraph goes through, except now every term is weighted by an extra factor of $\frac{1}{N^n}$. For example, the membrane like interaction produced by F_n looks like $\frac{1}{N^n}\frac{\alpha v^4}{\mu^2 z^5}$. This factor of $\frac{1}{N^n}$ means that this term is in fact a matrix theory correction to supergravity, because it vanishes as $N \to \infty$. Therefore, we could see that in converting the summation into a series of integrals, only the first one is needed for comparison with supergravity. All the other F_n with $n > 0$ produces only matrix theory corrections which is not the subject of interest here.

Now we go back to eqn (115). As discussed above, we only need the integral part F_0, which we calculate using Mathematica. After calculating the integrals, u is replaced by the supergravity variable $w = \frac{\alpha\mu}{3}u$, and the answer is expanded first in large N, keeping only the leading order (which is order N^0), and then expanded in small α, keeping only the $(\alpha)^1$ order (which according to Subsection 2.2 is the appropriate order to be compared to supergravity in the membrane limit). Finally the answer is converted back into Minkowski signature by sending $v^2 \to -v^2$. We obtain the following one-loop potential in the membrane limit[4]:

$$V_{eff} = \alpha\left(\frac{(36v^2 - z^2w^2)(108v^2 - (4w^2 + 7z^2)\mu^2)}{1152(w^2 + z^2)^{5/2}\mu^2}\right). \tag{120}$$

We find exact agreement when this expression is compared with the supergravity light cone Lagrangian given in eqn (95).

6 Interpolation of the Effective Potential

6.1 The Interpolation

The purpose of this section is to find the effective potential that interpolates between the membrane limit and the graviton limit. Due to the complexity of the field equations on the supergravity side, this problem could only be attacked on the matrix theory side. On the matrix theory side, however, there is a subtlety that needs to be taken into account before such a potential could be found. Here we will analyse only the v^4 term, the $\mu^2 v^2$ term as well as μ^4 term can be studied in exactly the same way.

From the supergravity side, what we wish to find is more or less clear. Near the membrane,

[4]This is only the effective potential in the membrane limit because when we expand in large N and keeping only the lowest order we essentially send the radius of the spheres $r_0 \sim \alpha\mu N$ to infinity, thus compare with z we have $\frac{z}{r_0} \ll 1$. Note also that the order the limits are taken is important. If the small α limit is taken before the large N limit, implicitly we would be assuming $\frac{z}{\alpha\mu} \gg N$ which is the graviton limit.

when $\frac{z}{r_0} \ll 1$, we expect an expansion like:

$$V_{eff} = \frac{\alpha v^4}{\mu^2 z^5}(1 + \frac{z}{r_0} + (\frac{z}{r_0})^2 + (\frac{z}{r_0})^3 + (\frac{z}{r_0})^4 + \cdots) \tag{121}$$

Far away from the membrane, when $\frac{z}{r_0} \gg 1$, we expect an expansion:

$$V_{eff} = \frac{\alpha v^4}{\mu^2 z^5}\frac{r_0^2}{z^2}(1 + \frac{r_0}{z} + (\frac{r_0}{z})^2 + (\frac{r_0}{z})^3 + (\frac{r_0}{z})^4 + \cdots) \tag{122}$$

We have used $\frac{N^2\alpha^3 v^4}{z^7} = \frac{\alpha v^4}{\mu^2 z^5}\frac{r_0^2}{z^2}$ to rewrite the graviton result so that it looks more similar to the membrane effective potential.

Therefore, we are basically looking for a function $\mathcal{C}(x)$ which appears in the effective potential in the following way:

$$V_{eff} = \frac{\alpha v^4}{\mu^2 z^5}\mathcal{C}(\frac{z}{r_0}) \tag{123}$$

As one could see, both the graviton and the membrane action is in this form. $\mathcal{C}(x)$ should have the appropriate limit at $x \to 0$ and $x \to \infty$ to give the correct potential at the membrane and the graviton limit respectively. Physically it represents curvature corrections due to the finite size of the spherical membrane.

So now we go to the matrix theory side to try to find $\mathcal{C}(x)$. The subtlety is that the effective potential on the matrix theory side not only includes curvature corrections but also the matrix theory corrections to supergravity which we are not interested in, not to mention that the sum over mass formula is incapable of deducing such matrix theory corrections exactly [2].

For our purpose of comparing with supergravity, therefore, all matrix theory corrections to supergravity should be thrown away. Such corrections appear on the matrix theory side as $1/N$ corrections, mixed together with the curvature corrections, and it looks something like:

$$V_{eff} = \frac{\alpha v^4}{\mu^2 z^5}\left(\mathcal{C}_0(\frac{z}{r_0}) + \frac{1}{N}\mathcal{C}_1(\frac{z}{r_0}) + \frac{1}{N^2}\mathcal{C}_2(\frac{z}{r_0}) + \cdots\right) \tag{124}$$

In such an expansion, only the $\mathcal{C}_0$ term should be kept. The readers are cautioned that naively sending N to infinity will not give us the correct interpolating potential because r_0 also depends on N and such limit would only result in the effective potential in the membrane limit.

There are many ways such matrix corrections could appear. For example, in a typical matrix theory computation we may get terms of the form:

$$V_{eff} \sim \frac{\alpha v^4}{\mu^2 z^7}(z^2 + \alpha^2\mu^2) \tag{125}$$

Rewriting the above gives:

$$V_{eff} \sim \frac{\alpha v^4}{\mu^2 z^5}(1+\frac{\alpha^2\mu^2 N^2}{z^2}\frac{1}{N^2}) \tag{126}$$

$$\sim \frac{\alpha v^4}{\mu^2 z^5}(1+\frac{r_0^2}{z^2}\frac{1}{N^2}) \tag{127}$$

The second term could now be identified as a matrix theory correction and is irrelevant to us.

To isolate the curvature corrections (which we want) from the matrix theory corrections (which we do not want), we look at eqn(124) more carefully. We could see that since $\frac{1}{N}=\frac{\alpha\mu}{6r_0}$, all the matrix theory corrections will appear in higher order in α. Therefore to get the interpolating effective potential from the matrix theory side, we could follow the steps below:

1. Change the summation over j in eqn(115) into an integral over j from 0 to N;
2. Replace N by $\frac{6r_0}{\alpha\mu}$, and u by $\frac{3w}{\alpha\mu}$;
3. Expand in small α and keeping only the lowest order (which shall turn out to be order α^1, with all lower orders vanishing). This is the interpolating effective potential.

With Mathematica, the interpolating effective potential for two spheres of the same radius ($w=0$) in Minkowski signature is found to be:

$$\begin{aligned} V_{eff} = {} & \frac{\alpha}{\mu^2 z^5}\frac{36v^2-z^2\mu^2}{1152(4r_0^2+z^2)^{5/2}} \\ & \times \Bigg\{ 108v^2\Big(-z^5+16r_0^4\sqrt{4r_0^2+z^2}+8r_0^2z^2\sqrt{4r_0^2+z^2}+z^4\sqrt{4r_0^2+z^2}\Big) \\ & -z^2\mu^2\Big(112r_0^4\sqrt{4r_0^2+z^2}+7z^4(-z+\sqrt{4r_0^2+z^2})+8r_0^2z^2(-2z+7\sqrt{4r_0^2+z^2})\Big)\Bigg\} \end{aligned} \tag{128}$$

We see that V_{eff} always carries a factor of $36v^2-z^2\mu^2$, meaning the effective potential vanishes whenever $v=\frac{\mu z}{6}$. This is expected because such configurations correspond to circular orbits which preserve half of the supersymmetries.

Expanding this potential in the membrane limit of large r_0 we get:

$$V_{eff} = \frac{\alpha(3888v^4-360v^2z^2\mu^2+7z^4\mu^4)}{1152\mu^2 z^5} = \frac{\alpha(36v^2-z^2\mu^2)(108v^2-7z^2\mu^2)}{1152\mu^2z^5} \tag{129}$$

This result is of course identical to the matrix theory result (with $w=0$) in section 5 where the membrane limit was taken in advance.

In the graviton limit of small r_0, after replacing r_0 by $\alpha\mu N/6$, we have:

$$V_{eff} = \frac{N^2\alpha^3(720v^4 - 56v^2z^2\mu^2 + z^4\mu^4)}{768z^7} = \frac{N^2\alpha^3(20v^2 - z^2\mu^2)(36v^2 - z^2\mu^2)}{768z^7} \tag{130}$$

The two limits of the effective action could then be compared with that of the supergravity side. Indeed from the expressions (96) and (97) we see that we have perfect agreement.

In above we have only given the expression of the interpolating potential for two membranes with the same radius ($w = 0$). We have also found the interpolating potential with w included, using the same steps given above. However we choose to omit the rather lengthy expression here for brevity.

6.2 Comparison with Shin and Yoshida

As mentioned in section 5, ref. [10], considered the case of supersymmetric circular motion with angular frequency $\frac{\mu}{6}$ and found a flat potential, which agrees with what we have found (see the comment under eqn(128)).

In a subsequent paper, [11], the authors considered the case of a slightly elliptical orbit with separation, z,

$$\begin{aligned} z &= \sqrt{(r_2+\epsilon)\cos^2\left(\frac{\mu t}{6}\right) + (r_2-\epsilon)\sin^2\left(\frac{\mu t}{6}\right)} \\ &= \sqrt{r_2^2 + \epsilon^2 + 2r_2\epsilon\cos\left(\frac{\mu t}{3}\right)} \end{aligned} \tag{131}$$

and velocity, v,

$$\begin{aligned} v &= \frac{\mu}{6}\sqrt{\left((r_2+\epsilon)\sin^2(\frac{\mu t}{6}) + (r_2-\epsilon)\cos^2\left(\frac{\mu t}{6}\right)\right)} \\ &= \frac{\mu}{6}\sqrt{\left((r_2^2 + \epsilon^2 - 2r_2\epsilon\cos\left(\frac{\mu t}{3}\right)\right)}. \end{aligned} \tag{132}$$

where ϵ is the small expansion parameter for the eccentricity of the orbit. They considered the large separation limit, $r_2 \gg 0$, and found an effective action, eqn(1.2) of [11],

$$\begin{aligned} \Gamma_{eff} &= \epsilon^4 \int dt\ (\alpha^3\mu^4N^2)\left(\frac{35}{2^7\cdot 3}\frac{1}{r_2^7} - \frac{385(\alpha\mu N)^2}{2^{11}\cdot 3^3}(4 - \frac{1}{N^2})\frac{1}{r_2^9}\right) \\ &= \frac{35}{384}\epsilon^4 \int dt\ (\alpha^3\mu^4N^2)\left(\frac{1}{r_2^7} - \frac{11(\alpha\mu N)^2}{36}(1 - \frac{1}{4N^2})\frac{1}{r_2^9}\right) \end{aligned} \tag{133}$$

after expanding to $O(\epsilon^4)$ and $O(1/r_2^9)$. Note that in the equation above we have restored μ and α, and set $N_1 = N_2 = N$, $r_1 = 0$. To compare this with our result we substitute

the above expressions of $z(t)$ and $v(t)$ into our interpolating potential eqn (128) and expand in the parameters $1/r_2$ and ϵ. As we are only comparing effective actions we average our potential over one period of oscillation. We find for our time-averaged potential,

$$\begin{aligned} V_{eff} &= \alpha\epsilon^4\mu^2\frac{105}{32}r_0^2\left(\frac{1}{r_2^7}-11\frac{r_0^2}{r_2^9}\right) \\ &= \frac{35}{384}\alpha^3\epsilon^4\mu^4N^2\left(\frac{1}{r_2^7}-\frac{11(\alpha\mu N)^2}{36}\frac{1}{r_2^9}\right) \end{aligned} \tag{134}$$

(where to reach the final line we used $r_0 = \frac{\alpha\mu N}{6}$) which agrees with eqn (133) after throwing away the $\frac{1}{N^2}$ matrix theory corrections in the latter. In calculating the matrix theory effective potential in section 5 we assumed a constant velocity. However, as we can see by this comparison, as long as we ignore matrix theory corrections, it leads to the correct result. Thus we see that we can consistently neglect acceleration terms in the effective potential as discussed earlier.

7 Discussion and Future Directions

One immediate generalization of the work reported in this paper is to consider more complicated probe configurations (recall that in this paper we have restricted our attention to a probe membrane which is spherical and has no velocity in the x^1 through x^3 directions). For example, we could consider deforming the probe membrane so that it is no longer a perfect sphere. It means that the coordinates of the membrane X^A will now be some general functions of θ and ϕ, and in particular it no longer has to appear only as a point in the 4 through 9 directions. We can also give the probe membrane a nonzero velocity in the x^1 through x^3 direction. These generalizations give more interesting dynamics and can be fairly easily carried out. On the supergravity side, this just requires putting the relevant probe configuration into the light-cone Lagrangian; on the matrix theory side, this requires replacing the background configurations $B^A_{(2)}$'s used in this paper with some more general configurations.

In this paper we take the source to be a static spherical membrane preserving all of the sixteen linearly realized supersymmetries. As we know, there are other interesting membrane configurations that are permitted by the pp-wave background, for example, the static hyperbolic brane which preserves eight supersymmetries, given by [19]. It would be interesting to investigate gauge/gravity duality in the case where these static objects are sources and interact with some graviton or membrane probe. On the supergravity side, we can apply our general formalism of diagonalizing the field equations as well as the near-membrane expansion to find the metric and three-form perturbations produced by these sources. On the

matrix theory side, for the aforementioned hyperbolic membrane, one could use the unitary representations of $SO(2,1)$ worked out in [20, 21, 22] to expand the fluctuating fields and compute the one-loop effective potential.

Moreover, it is evidently desirable to push our computation on the supergravity side beyond the near-membrane expansion, finding the solution to the field equations which will then give the $\delta\mathcal{L}_{lc}$ to be compared with the fully interpolating potential (128) found on the matrix theory side.

The membrane-interaction we described in this paper has no longitudinal momentum transfer. However, as we can see, our supergravity computation can be quite easily generalized to nonzero longitudinal momentum transfer, by not setting k_- to zero when solving the field equations (e.g., see the comments made after eqn. (77)). On the gauge theory side, this requires an instanton compuation. We hope to say more on this in the near future.

There are also M-theory pp-wave backgrounds with less supersymmetries, and matrix theories in these pp-waves backgrounds have been proposed [23, 24]. It would be interesting to investigate the gauge/gravity duality in these less supersymmetric settings. Quite recently a matrix theory dual of strings on the maximally supersymmetric ten-dimensional IIB pp-wave background was conjectured by [25]. This matrix theory is the regularization of the action of D3-branes (tiny gravitons). It would be very interesting to investigate this conjecture along the lines of this paper and [2].

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